REFERENCES

1. Aizerman, M. A. and Gantmakher, F. R., Absolute Stability of Controlled Systems. Moscow, Izd. Akad. Nauk SSSR, 1963.
2. Gantmakher, F. R. and Iakubovich, V. A., Absolute stability of nonlinear controlled systems. Proc. of 2nd All-Union Conference on Theoretical and Applied Mechanics. Moscow, "Nauka", 1965.
3. Iakubovich, V. A., Solution of certain matrix inequalities encountered in the theory of automatic control. Dokl. Akad. Nauk SSSR, Vol. 143, № 6, 1962.
4. Iakubovich, V. A., Absolute stability of nonlinear controlled systems in the critical cases. III. Avtomatika i telemekhanika, Vol. 25, № 5, 1964.
5. Iakubovich, V. A., Periodic and almost periodic limiting modes of controlled systems with several, generally speaking, discontinuous nonlinearities. Dokl. Akad. Nauk SSSR, Vol. 171, № 3, 1966.
6. Kalman, R. E., Liapunov functions for the problem of Lur'e in automatic control. Proc. Nat. Acad. Sci. USA, Vol. 49, 1963.
7. Popov, V. M., Hyperstability of Automatic Systems. Moscow, "Nauka", 1970.
8. Iakubovich, V. A., Solution of an algebraic problem encountered in the control theory. Dokl. Akad. Nauk SSSR, Vol. 193, № 1, 1970.
9. Lions, J. L., Optimal Control of Systems Governed by Partial Differential Equations. N. Y., Berlin, Springer-Verlag, 1971.
10. Andreev, V. A., Kazarinov, Iu. F. and Iakubovich, V. A., Synthesis of optimal controls for linear inhomogeneous systems in problems of minimizing quadratic functionals. Dokl. Akad. Nauk SSSR, Vol. 199, № 2, 1971.
11. Yosida, K., Functional Analysis. Springer-Verlag, Berlin, Heidelberg, N. Y., 1974.
12. Barsuk, L. O. and Brusin, V. A., Infinitely dimensional generalization of the Kalman-Iakubovich lemma. In coll.: The Dynamics of Systems. 8th Ed. Izd. Gor'kii Univ., 1975.

Translated by L. K.

UDC 539.3

ON THE CONSTRUCTION OF GENERAL SOLUTIONS OF THE ELASTICITY THEORY EQUATIONS FOR TRANSVERSELY ISOTROPIC INHOMOGENEOUS BODIES

PMM Vol. 40, № 5, 1976, pp. 956-958

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(Received April 29, 1975)

The solution is presented for the three-dimensional problem of the theory of elasticity of transversely isotropic elastic bodies, where the elastic characteristics vary arbitrarily along the axis of symmetry of the elastic properies of the medium. The solution is written in orthogonal curvilinear cylindrical coordinates and is represented by using two independent functions. The question of separation of the boundary conditions in the plane of isotropy is examined.

A number of investigations, which examine primarily the equilibrium of isotropic bodies with an exponential law of variation of the elastic modulus and a constant Poisson's ratio, is devoted to the solution of two-dimensional problems of the theory of elasticity of inhomogeneous bodies. One of the first works in this area is apparently [1]. Under analogous assumptions, the general solution of the three-dimensional problem of the theory of elasticity of transversely isotropic bodies and isotropic bodies adapted for the analysis of laminar media has been constructed in [2]. It is also represented by using two independent functions for which the conditions of the boundaries of the layers are separated. The dependences obtained are used in [2] for the general solution of the equilibrium problem for half-spaces comprised of layers which are not homogeneous with depth under the effect of surface forces. A solution of the three-dimensional problem of the theory of elasticity of an inhomogeneous isotropic body, constructed by a scheme similar to that elucidated, but without the constraints imposed on the elastic characteristics and taking account of the volume forces is presented in [3].

Following Gutman [4], let us represent the required solution as the sum of components of the first and second kinds. In the solution of the first kind, defined by the function Π, ω_z vanishes, while in the solution of the second kind defined by the function Ψ, the deflection w, the stress σ_z and the volume expansion vanish. We have

$$u_\alpha = \frac{1}{H_1}\frac{\partial F}{\partial \alpha} + \frac{1}{H_2}\frac{\partial L}{\partial \beta}, \quad u_\beta = \frac{1}{H_2}\frac{\partial F}{\partial \beta} - \frac{1}{H_1}\frac{\partial L}{\partial \alpha} \tag{1}$$

$$F = \beta_{33} D^2\Pi + \beta_{11}\frac{\partial^2 \Pi}{\partial z^2}, \quad L = \frac{1}{G}\frac{\partial \Psi}{\partial z}$$

$$w = -\frac{1}{G_1}\frac{\partial}{\partial z} D^2\Pi - \frac{\partial}{\partial z}\left(\beta_{13} D^2\Pi + \beta_{11}\frac{\partial^2\Pi}{\partial z^2}\right)$$

$$\tau_{\alpha z} = \frac{1}{H_1}\frac{\partial \tau}{\partial \alpha} + \frac{1}{H_2}\frac{\partial s}{\partial \beta}, \quad \tau_{\beta z} = \frac{1}{H_2}\frac{\partial \tau}{\partial \beta} - \frac{1}{H_1}\frac{\partial s}{\partial \alpha} \tag{2}$$

$$\tau = -\frac{\partial}{\partial z} D^2\Pi, \quad s = -D^2\Psi, \quad \sigma_z = D^4\Pi$$

$$D^2 = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial \alpha}\left(\frac{H_2}{H_1}\frac{\partial}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{H_1}{H_2}\frac{\partial}{\partial \beta}\right)\right]$$

$$\beta_{11} = \frac{1-\nu^2}{E}, \quad \beta_{13} = -\frac{\nu_1(1+\nu)}{E_1}, \quad \beta_{33} = \frac{1}{E_1}\left(1 - \frac{\nu_1^2 E}{E_1}\right), \quad G = \frac{E}{2(1+\nu)}$$

Here α, β are curvilinear coordinates in the isotropy plane, the z-axis coincides with the axis of symmetry of the medium, H_1, H_2 are Lamé coefficients, β_{11}, β_{13}, β_{33} are the reduced elastic constants introduced by S. G. Lekhnitskii, E, ν are the elastic modulus and Poisson's ratio in the plane of anisotropy, E_1 is the elastic modulus in the z-direction, ν_1 is the Poisson's ratio taking account of the influence e_z on the strain in the isotropy plane, and G_1 is the shear modulus in a plane perpendicular to the plane of isotropy.

The functions Π and Ψ satisfy the equations

$$\beta_{33} D^4\Pi + D^2\left\{\frac{\partial}{\partial z}\left[\left(2\beta_{13} + \frac{1}{G_1}\right)\frac{\partial \Pi}{\partial z}\right] + \Pi\frac{\partial^2 \beta_{13}}{\partial z^2}\right\} + \frac{\partial^2}{\partial z^2}\left(\beta_{11}\frac{\partial^2 \Pi}{\partial z^2}\right) = 0 \tag{3}$$

$$D^2\Psi + G_1\frac{\partial}{\partial z}\left(\frac{1}{G}\frac{\partial \Psi}{\partial z}\right) = 0$$

Further, only the strains defined by differentiation in the plane of isotropy by means of the formulas

$$e_\alpha = \frac{1}{H_1}\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\beta}{H_1 H_2}\frac{\partial H_1}{\partial \beta}, \quad e_\alpha + e_\beta = D^2 F$$

$$\gamma_{\alpha\beta} = \frac{H_1}{H_2}\frac{\partial}{\partial \beta}\left(\frac{u_\alpha}{H_1}\right) + \frac{H_2}{H_1}\frac{\partial}{\partial \alpha}\left(\frac{u_\beta}{H_2}\right)$$

are expressed analytically.

The stresses σ_α, σ_β, $\tau_{\alpha\beta}$ and the strain e_z are calculated by means of Hooke's law formulas which are converted into

$$\sigma_\alpha = \alpha_{11} e_\alpha + \alpha_{12} e_\beta + \alpha_{13}\sigma_z, \quad \sigma_\beta = \alpha_{12} e_\alpha + \alpha_{11} e_\beta + \alpha_{13}\sigma_z$$

$$e_z = \alpha_{31}(e_\alpha + e_\beta) + \alpha_{33}\sigma_z$$

$$\alpha_{11} = \frac{E}{1-\nu^2}, \quad \alpha_{12} = \nu\alpha_{11}, \quad \alpha_{13} = -\alpha_{31} = \frac{\nu_1 E}{E_1(1-\nu)}$$

$$\alpha_{33} = \frac{1}{E_1}\left[1 - \frac{2\nu_1^2 E}{(1-\nu)E_1}\right]$$

Let us show that the solutions of the first and second kinds can be examined independently for a layer bounded by isotropy planes. The same refers to a multilayered half-space.

The equations to determine the functions Π and ψ are separated. The boundary values of the functions σ_z and τ (the first static problem) or of w and F (the second static problem) are used in the solution of the first kind, and boundary values of the function s (the first statics problem) or L (the second statics problem) in the solution of the second kind.

To separate the solutions completely it is required to determine the boundary values of the function τ and s by the quantities τ_0, s_0 or the quantities F_0, L_0. For example, we have $\sigma_z = p$, $\tau_{\alpha z} = t_1$, $\tau_{\beta z} = t_2$ at $z = h$ for the first statics problem.

In conformity with (2), let us assume

$$t_1 = \frac{1}{H_1}\frac{\partial \tau_0}{\partial \alpha} + \frac{1}{H_2}\frac{\partial s_0}{\partial \beta}, \quad t_2 = \frac{1}{H_2}\frac{\partial \tau_0}{\partial \beta} - \frac{1}{H_1}\frac{\partial s_0}{\partial \alpha}$$

from which it follows

$$D^2\tau_0 = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial \alpha}(H_2 t_1) + \frac{\partial}{\partial \beta}(H_1 t_2)\right] \tag{4}$$

$$D^2 s_0 = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial \beta}(H_1 t_1) - \frac{\partial}{\partial \alpha}(H_2 t_2)\right]$$

The boundary values of the functions F and L (the second statics problem) are written in an analogous manner.

The conditions for continuity of the stresses $\tau_{\alpha z}$, $\tau_{\beta z}$ and displacement u_α, u_β in the computation of a laminar half-space are equivalent to the continuity conditions for the functions F and τ (solution of the first kind) or s, L (solution of the second kind).

It also follows from the above that the problem of analyzing an inhomogeneous layer or half-space is partitioned in two independent stages, where the invariant functions σ_z, w, τ, F, s, L are determined in the first stage. An analogy is noted between the first stage of the solution of two-dimensional (plane and axisymmetric) problems about the equilibrium of an inhomogeneous half-space and the three-dimensional problem written in an arbitrary orthogonal curvilinear coordinate system. This analogy is also detected between solutions of axisymmetric problems for a continuous layer (half-space) and a layer (half-space) with an absolutely stiff and smooth cylindrical inclusion.

In fact, the kernels of the integral transforms used in solving the problems mentioned satisfy the equation

$$D^2 f = -\gamma^2 f \tag{5}$$

It follows from (2), (4), (5) that the algorithms to determine the transformant of the invariant functions σ_z, τ, w, F agree in the cases mentioned. Therefore, the most tedious part of the solution can be used in examining a number of problems. The same refers to the solutions of the second kind represented in Cartesian and curvilinear coordinates.

The inversion formulas, expressions for the load transformant and results of a computation are understandably distinct for analogous problems.

The order taken to analyze inhomogeneous bodies is also used in a variant of the finite strip method (interpolation method) proposed in [5, 6] for the consideration of the multilayered bodies, and contributes to shortening the computational operations.

REFERENCES

1. Ter-Mkrtich'ian, L. N., Some problems in the theory of elasticity of inhomogeneous elastic media. PMM Vol. 25, № 6, 1961.
2. Rappoport, R. M., Equilibrium of a half-space comprised of layers which are inhomogeneous in depth (three-dimensional problem). Trudy Leningrad Wood Technology Academy, № 109, 1967.
3. Plevako, V. P., Inhomogeneous layer bonded to a half-space under the action of internal and external forces. PMM Vol. 38, № 5, 1974.
4. Gutman, S. G., General solution of elasticity theory problems in generalized cylindrical coordinates. Izv. B. E. Vedeneev All-Union Scient.-Research Inst. of Hydraulic Engineering, Vol. 37, , 1948.
5. Rappoport, R. M., On the question of constructing solutions of axisymmetric and plane problems of the theory of elasticity of a multilayered medium. Izv. B. E. Vedeneev All-Union Sci.-Res. Inst. Hydraulic Eng'ng., Vol. 73, 1963.
6. Rappoport, R. M., Interpolation solutions of the theory of laminar slab bending. Analysis of Spatial Structures, № 14, Stroiizdat, Moscow, 1971.

Translated by M. D. F.

ON THE PROBLEM OF INTERACTION OF RESONANCES

PMM Vol. 40, № 5, 1976, pp. 959-960

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(Received October 16, 1975)

The stability is studied of the neutral equilibrium of a system, in a linear approximation, which has two resonances locked at two frequencies, each of which does not cause instability separately in a second approximation. It is shown that in contrast to the case of independent resonances and those locked at one frequency, stability can be lost (in the same order).